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AN IMPROVED TRANSFORMATION OF THE PATANKA – SPALDING TYPE FOR NUMERICAL SOLUTION OF TWO-DIMENSIONAL BOUNDARY LAYER FLOWS

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INTRODUCTION

Substantial impetus has been given to the numerical solution of two-dimensional boundary layer flows by the work of Patankar and Spalding [1, 2]. The primary contribution consists of a coordinate transformation (hereafter, the "ω-transformation") from the physical variables (x, y) to variables (x, ω), where

$$\omega = \int_0^y \rho u \mathrm{d}y \Big/ \int_0^{y_E} \rho u \mathrm{d}y = (\psi - \psi_I)/(\psi_E - \psi_I) \tag{1}$$

and the subscripts I and E denote, respectively, the inner and outer edges of the boundary layer. Use of the non-dimensional stream function ω as the cross-stream variable confines the boundary layer to the rectangular region $x > 0$, $0 \leqslant \omega \leqslant 1$. This fact coupled with a working expression for mass entrainment, $\mathrm{d}\psi_E/\mathrm{d}x$, results in efficient utilization of the grid network in a finite difference solution.

The present authors have encountered some unexpectedly large errors in results obtained with this formulation, particularly when finite difference analogues of the governing differential equations are derived from Taylor series expansions. This is the result of inherent inaccuracies in finite difference approximations in the near wall region since

$$\partial u/\partial \omega = \frac{\partial u}{\partial y}\frac{\partial y}{\partial \omega} = \frac{\Delta\psi}{\rho u}\frac{\partial u}{\partial y}, \tag{2}$$

and all higher derivatives of u with respect to ω, become infinite as $y \to 0$. The resulting higher truncation errors at node points placed near the wall is of primary concern due to its effect on the extraction of wall gradients.

Patankar and Spalding circumvented this difficulty by matching a Couette flow analysis in the near wall region with the finite difference solution away from the wall. Although this is undoubtedly satisfactory for most applications, the complexity of the concept combined with the need to derive new expressions for each class of problems treated has resulted in rejection of this feature by many authors [3–5]. In addition, Patankar and Spalding recommend that difference analogues be obtained by integrally averaging the conservation equation over a control volume extending from $\omega_{i-\frac{1}{2}}$ to $\omega_{i+\frac{1}{2}}$ with an assumed linear variation of the dependent variable between adjacent node points. As will be shown here, this serves to reduce near wall truncation errors associated with the ω-transformation.

The purpose of this note is to present a modified transformation (hereafter, the "ω^2-transformation")

$$\omega^2 = \frac{\int_0^y \rho u \mathrm{d}y}{\int_0^{y_E} \rho u \mathrm{d}y} \tag{3}$$

which is similar to that of Patankar and Spalding in its implementation, but yields more accurate numerical results due to the well-behaved nature of the solution near the wall. In addition, a method for extracting wall gradients directly in the ω-plane is discussed. Numerical results are obtained for constant property, zero pressure gradient flows over a semi-infinite flat plate with self-similar blowing or suction at the wall.

ANALYSIS

Combining equations (1) and (3) in the general form $\omega^n = (\psi - \psi_I)/\Delta\psi$, the continuity and momentum equations for two dimensional boundary layer flow on a flat plate

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{4}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\, \mu_{\mathrm{Eff}} \frac{\partial u}{\partial y} \tag{5}$$

become [6]

$$\frac{\partial u}{\partial x} - \frac{1}{n\Delta\psi\omega^{n-1}}\left(\frac{\mathrm{d}\psi_I}{\mathrm{d}x} + \omega^n \frac{\mathrm{d}\Delta\psi}{\mathrm{d}x}\right)\frac{\partial u}{\partial \omega} = \frac{1}{n\Delta\psi\omega^{n-1}}\frac{\partial}{\partial\omega}\left(\frac{\rho u \mu_{\mathrm{Eff}}}{n\Delta\psi\omega^{n-1}}\frac{\partial u}{\partial\omega}\right). \tag{6}$$

For comparative purposes, difference analogues of (6) were obtained by two methods:

Case I. *Integrally averaged difference analogue*

Following [1], an approximate form of (6) may be obtained by integrating over a control volume bounded by x_j, x_{j-1} and $(\omega_i + \omega_{i+1})/2$, $(\omega_i + \omega_{i-1})/2$

$$\frac{2}{(x_j - x_{j-1})(\omega_{i+1} - \omega_{i-1})}\int_{x_{j-1}}^{x_j}\int_{(\omega_{i-1}+\omega_i)/2}^{(\omega_i+\omega_{i+1})/2}\left[n\Delta\psi\omega^{n-1}\frac{\partial u}{\partial x} - \left(\frac{\mathrm{d}\psi_I}{\mathrm{d}x} + \omega^n\frac{\mathrm{d}\Delta\psi}{\mathrm{d}x}\right)\frac{\partial u}{\partial\omega} - \frac{\partial}{\partial\omega}\left(\frac{\rho u\mu_{\mathrm{Eff}}}{n\Delta\psi\omega^{n-1}}\frac{\partial u}{\partial\omega}\right)\right]\mathrm{d}\omega\mathrm{d}x = 0 \tag{7}$$

and assuming that u varies linearly with ω between cross-stream grid points.

Case II. *Central difference analogue*

Employing implicit central difference expressions for the cross-stream derivatives and a backward difference expression for the streamwise derivative yields

$$\frac{u_{i,j} - u_{i,j-1}}{x_j - x_{j-1}} - \frac{1}{n\Delta\psi_{j-1}\omega_i^{n-1}}\left(\left.\frac{\mathrm{d}\psi_I}{\mathrm{d}x}\right|_{j-1} + \omega_i^n\left.\frac{\mathrm{d}\Delta\psi}{\mathrm{d}x}\right|_{j-1}\right) \times \left(\frac{\delta_1}{\delta_{12}}\frac{u_{i+1,j} - u_{i,j}}{\delta_2} + \frac{\delta_2}{\delta_{12}}\frac{u_{i,j} - u_{i-1,j}}{\delta_1}\right)$$
$$\doteq \frac{1}{n\Delta\psi_{j-1}\omega_i^{n-1}}\frac{2}{\delta_{12}}\left[\left.\frac{\rho\mu_{\mathrm{Eff}}u}{n\Delta\psi\omega^{n-1}}\right|_{i+\frac{1}{2},j-1}\frac{u_{i+1,j} - u_{i,j}}{\delta_2} - \left.\frac{\rho\mu_{\mathrm{Eff}}u}{n\Delta\psi\omega^{n-1}}\right|_{i-\frac{1}{2},j-1}\frac{u_{i,j} - u_{i-1,j}}{\delta_1}\right] \tag{8}$$

where $\delta_1 = \omega_i - \omega_{i-1}$, $\delta_2 = \omega_{i+1} - \omega_i$, and $\delta_{12} = \omega_{i+1} - \omega_{i-1}$.

Both analogues result in sets of linearized algebraic equations in the form

$$u_{i,j} = A_{i,j-1}u_{i+1,j} + B_{i,j-1}u_{i-1,j} + C_{i,j-1}, i = 2,3\ldots N \tag{9}$$

which can be solved by efficient elimination methods [1].

For illustrative purposes, the problem considered here is that of constant property laminar boundary layer flow with self-similar blowing or suction at the wall. The flow is governed by equation (6) with $\mu_{\mathrm{Eff}} \to \mu$ and is subject to the boundary conditions

$$\begin{aligned} &u = 0 \\ &\rho v = -\frac{\mathrm{d}\psi_I}{\mathrm{d}x} = -f_I\,\rho u_\infty/2Re_x^{\frac{1}{2}} \qquad \text{at } y = 0 \\ &u = u_\infty \qquad \text{as } y \to \infty \end{aligned} \tag{10}$$

where f_I is a constant blowing/suction parameter [7].

For the self-similar problem, the entrainment law can be replaced by the exact result

$$\mathrm{d}\Delta\psi/\mathrm{d}x = (\rho u_E/2Re_x^{\frac{1}{2}})\int_0^{\eta_E}(u/u_\infty)\,\mathrm{d}\eta. \tag{11}$$

Also, $u_E = u(\eta_E)$. This eliminates the minor errors incurred in the Patankar–Spalding formulation when (i) applying the external boundary condition $u = u_\infty$ at $\omega = 1$, and (ii) using approximate entrainment rates.

RESULTS AND DISCUSSION

Numerical solutions of equation (6), subject to the boundary conditions (10), were effected for the physical parameters $f_I = -0.75$ and 5.0. $u_\infty = 100$ ft/s, $\mu = 10^{-5}$ lb/fts, and $\rho = 0.075$ lb/ft^3. Results were obtained using 25, 50 and 100 node points for combinations of transformation (ω and ω^2) and difference analogue (integral averaged and central difference). In addition, two choices for "stacking" node point distributions $\{\omega_i\}$, as extracted from initial y-distributions

$$y_i = [(i-1)/N]^{\gamma} y_E, \qquad \gamma = 1 \text{ and } 2 \tag{12}$$

were used. The numerical solutions were advanced step-wise until $c_f Re_x^{\frac{1}{2}}$ was unchanging to six significant figures. Percentage errors in u_i and c_f were computed by means of comparisons with "exact" self-similar results. In calculating numerical values of c_f, wall gradients were obtained by differentiating third order polynomial fits to the near wall data using

$$\frac{\partial u}{\partial y} = \frac{\rho u}{\Delta\psi}\frac{\partial u}{\partial \omega} = \frac{\rho}{2\Delta\psi}\frac{\partial u^2}{\partial \omega} \quad \text{for } n = 1 \tag{13}$$

and

$$\frac{\partial u}{\partial y} = \frac{\rho u}{\Delta\psi\omega}\frac{\partial u}{\partial \omega} = \frac{\rho}{\Delta\psi}\left(\frac{\partial u}{\partial \omega}\right)^2 \quad \text{for } n = 2. \tag{14}$$

The present authors recommend that wall gradients be extracted directly in the ω-plane, as in equations (13) and (14), since the accuracy of the results obtained is consistent with that of the near wall numerical data.

Typical results are summarized in Tables 1 and 2. In general, the ω^2-transformation yields improved accuracies in the near wall region, particularly for suction (Table 2), where an order of magnitude improvement is obtained for both u_2 and c_f. Furthermore, it is seen that the ω^2-transformation is relatively insensitive to methods of differencing equation (6) (contrast Cases I and II for $n = 2$); such is not the case for the ω-transformation, where use of central difference analogues results in large errors. The success of the integrally averaged approach, as recommended in [1], is related to the extent to which the difference analogue for the cross-stream advective term is centered above the "2"-point. This conclusion is supported by the order of magnitude improvement in the Case I results when $\gamma = 2$ ($y_3 = 4y_2$ and $\omega_3 \doteq 16\omega_2$) vs. $\gamma = 1$ ($y_2 = 2y_2$ and $\omega_3 \doteq 4\omega_2$). Furthermore, as may be seen from Fig. 1, relatively larger errors are incurred, in the case of the ω-transformation, as $y_i/y_E \to 0$. Recalling the implications of equation (2), this is not surprising.

Table 1. *Results for* $f_1 = -0{\cdot}75$

Case	N	γ	% error $u_2\|_{n=1}$	% error $u_2\|_{n=2}$	% error $c_f\|_{n=1}$	% error $c_f\|_{n=2}$
I	25	1	−0·80	−0·043	2.84	−0·17
I	50	1	−0·45	−0·011	1.30	−0·024
I	100	1	−0·24	−0·0028	0·62	−0·0042
I	25	2	−0·12	−0·15	0·50	−0·30
I	50	2	−0·031	−0·038	0·13	−0·075
I	100	2	−0·0077	−0·0094	0·031	−0·019
II	25	1	−6·67	−0·079	−11·90	−0·23
II	50	1	−3·50	−0·020	−6·26	−0·041
II	100	1	−1·77	−0·0049	−3·17	−0·0084
II	25	2	−4·47	−0·12	−8·69	−0·24
II	50	2	−1·38	−0·030	−2·72	−0·061
II	100	2	−0·41	−0·0077	−0·81	−0·015

Case I —Integrally averaged analogue [equation (7)].
Case II —Central difference analogue [equation (8)].
$n = 1$ — ω-transformation.
$n = 2$ — ω^2-transformation.

Table 2. *Results for* $f_1 = 5{\cdot}0$

Case	N	γ	% error $u_2\|_{n=1}$	% error $u_2\|_{n=2}$	% error $c_f\|_{n=1}$	% error $c_f\|_{n=2}$
I	25	1	1·96	0·22	−4·03	0·64
I	50	1	0·96	0·056	−2·10	0·16
I	100	1	0·47	0·014	−1·07	0·040
I	25	2	0·071	−0·0047	−0·33	−0·011
I	50	2	0·018	−0·0013	−0·083	−0·0027
I	100	2	0·0044	−0·00034	−0·021	−0·00067
II	25	1	10·9	−0·0067	18·3	0·14
II	50	1	5·84	−0·0027	10·1	0·035
II	100	1	3·05	−0·0081	5·35	0·0088
II	25	2	7·35	0·16	15·1	0·33
II	50	2	2·29	0·041	4·61	0·082
II	100	2	0·69	0·010	1·38	0·020

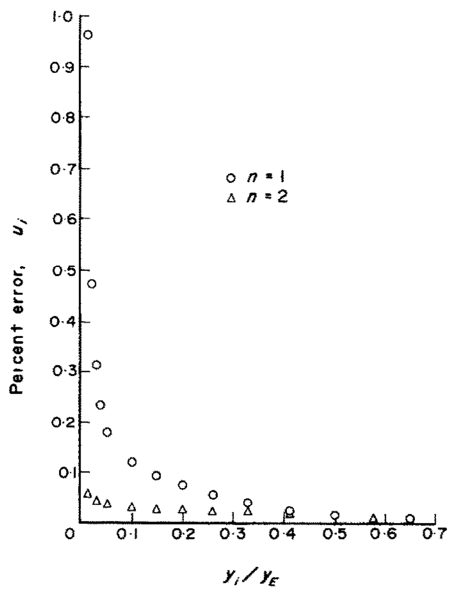

FIG. 1. Comparisons of errors in the near wall data for case I: $n = 1$(o) vs. $n = 2$(Δ) for $N = 50$ and $\gamma = 1$.

The results also indicate extraordinary sensitivity to node-point distribution, especially near the wall, and suggest that considerable improvement in accuracy, regardless of solution method, would accrue if an "optimal" distribution could be found. In this respect, the authors have developed a new method for two-point boundary value problems [8] which shows promise for marching problems as well.

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A COMMENT ON THE PERIODIC FREEZING AND MELTING OF WATER

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INTRODUCTION

LOCK *et al.* [1] have presented results for the periodic freezing and melting of water in an essentially undimensional cartesian arrangement wherein the temperature at the plane $x = 0$ varied in an approximately sinusoidal manner, with the mean value being the fusion temperature. The system initially was liquid so that freezing occurred in the first half period, with melting in the second half period, and the two hour period was executed successively thereafter. Figure 1 shows by points some of the results in terms of the depth in centimeters and the time in hours, with the points being taken from the first, second and fourth freeze and the first and third melt. Lock *et al.* showed that a good degree of correspondence was achieved both by approximate analysis and numerical solution of the conduction equation for the developing phase; the temperature in the existing phase being the fusion value. The analytic solution is not easy to evaluate and the numerical solution appears to have been prodigal of computer time.

In the water–ice system, the latent heat of fusion is large enough so that when the surface temperature is not far from the saturation value then the thermal capacity effects in the solid can be ignored (low Stephan number). Then the solution [2] of the problem involves conduction effects alone and is quite simple. Conversely, tests with water cannot really verify the degree to which a theory adequately accounts for the thermal capacity effects unless the surface temperature amplitude is made very large.

It is the present purpose to show that the simple theory adequately predicts the experimental features of Lock's results to the extent that the more complicated analysis does so. Lock has, in fact, already done this in a prior reference [3]. Also, there is examined his suggestion that some of the difference between theory and experiment is ascribable to the effect of convection as that has been indicated by the melting experiments of Yen [4]. This consideration shows that while the convective effect probably did exist and is in the direction required, it is barely discernible in terms of the results of the Lock experiments. Generally, however, the effect is important and should be considered in melting problems.